1. Let X_1 and X_2 be independent and identically distributed (iid) with probability density function (pdf)

$$f_X(x|\sigma^2) = \left(\frac{1}{\sigma^2}\right)^{\frac{1}{2}} \sqrt{\frac{2}{\pi}} e^{-x^2/2\sigma^2}$$

for $x > 0$ and $\sigma > 0$.

(a) What is the pdf X_1/X_2? Are X_1/X_2 and X_2 independent? Why or why not?

(b) Let σ^2 have an Inverted-Gamma(α, β) distribution. The pdf is

$$f_{\sigma^2}(\sigma^2) = \frac{1}{\Gamma(\alpha)\beta^\alpha} \left(\frac{1}{\sigma^2}\right)^{\alpha+1} e^{-1/\beta\sigma^2},$$

where $\sigma^2, \alpha, \beta > 0$. What is the distribution of σ^2 given $X_1 = x_1$? What is the distribution of σ^2 given $X_1 = x_1$ and $X_2 = x_2$?

2. Let X have the double exponential distribution with parameters μ and σ:

$$f_X(x|\mu, \sigma) = \frac{1}{2\sigma} e^{-|x-\mu|/\sigma}$$

where $-\infty < x, \mu < \infty$ and $\sigma > 0$. The mean of this distribution is μ and the variance is $2\sigma^2$.

(a) Show that the moment generating function (mgf) of X is

$$M_X(t) = \frac{e^{\mu t}}{1 - (\sigma t)^2}$$

for $|t| < 1/\sigma$. Hint: divide the integral at μ:

$$\int_{-\infty}^{\infty} \cdots dx = \int_{-\infty}^{\mu} \cdots dx + \int_{\mu}^{\infty} \cdots dx$$

where $\cdots$ is something you supply.

(b) Let μ have a Normal(θ, τ^2) distribution and σ^2 have an Inverted-Gamma(α, β) distribution (as in 1b), independent of one another. The mean and variance of the distribution of σ^2 are $[(\alpha - 1)\beta]^{-1}$ and $[(\alpha - 1)^2(\alpha - 2)\beta^2]^{-1}$, respectively. Assuming that the distribution of X given μ and σ is double exponential, what are the (unconditional) mean and variance of X?

3. Let $X_1, \ldots, X_n$ be a random sample form an Inverted-Gamma(α, β) distribution where α is known. Recall from before the pdf of an Inverted-Gamma(α, β) distribution is

$$f_X(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha} \left(\frac{1}{x}\right)^{\alpha+1} e^{-1/\beta x}$$

for $x, \alpha, \beta > 0$.

(a) Derive the MLE estimator of β.

(b) Derive the 0.05 level LRT test of $H_0 : \beta = \beta_0$ versus $H_1 : \beta > \beta_0$.

4. A two factor model based on the correlation matrix results in the following factors. Assume the PC method is used to estimate the loadings.

   (a) Interpret the meaning of factor 2 (Note: Gaelic=Irish language)

Gaelic	0.4	0.6
English	0.4	0.5
History	0.2	0.5
Arithmetic	0.8	-0.01
Algebra	0.8	-0.1
Geometry	0.6	-0.1

   (b) Briefly describe two different ways to assess the fit of the single factor model.

5. In an experiment, there are six soybean fields. The fields are assigned to different fertilizers. The table below shows the grain yield measured for each of the fields in metric tons/hectare.

Field	Fertilizer	Grain yield in mt/he
1	1	3.98
2	1	3.27
3	1	3.62
4	2	4.00
5	3	4.30
6	3	4.27

   Assume the treatments effects model for the described example, where μ represents the overall mean and the different α terms represent the fertilizer effects.

   (a) Write out the model in matrix form $\boldsymbol{Y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{e}$.

   (b) Derive the projection matrix. Show your work.

   (c) Which of the linear combinations of estimators are estimable, when there are no restrictions of any of the parameters? Explain. In the case that a combination is estimable, interpret its meaning (in plain English words).

      i. μ

      ii. $\mu + \alpha_2$

      iii. α_1

      iv. $\alpha_2 - \alpha_1$

   (d) Pick a set of estimable functions, which readily allows the comparison of the fertilizers, and write out the corresponding full rank matrix $\boldsymbol{X}^*$ together with the new parameter vector $\boldsymbol{\beta}^*$.

   (e) State and explain properties of a BLUE estimator of an element of $\boldsymbol{\beta}^*$.
   Sketch a proof of the properties.

   (f) In a larger multi-state experiment, a larger number of soybean fields was evaluated for 6 years. For fertilizer 1 25 fields were evaluated, and 27 fields were evaluated for both fertilizer 2 and 3. We want to compare two models:

i. Model A: $y_{itj} = \mu + \alpha_i + \beta t + e_{itj}$

ii. Model B: $y_{itj} = \mu + \alpha_i + \beta_i t + e_{itj}$

with μ the expected average grain yield, α_i are the fertilizer-specific effects ($i = 1, 2, 3$; $t = 1, 2, 3, 4, 5, 6$; $j = 1, \ldots, 25, 26, 27$, respectively). We assume that the grain yield is normally distributed.

i. Draw a sketch of expected grain yield versus time for each model.

ii. Explain the difference between the models.