Each question scored on 10
with each point given equal wt.

1. Let X_1 and X_2 be independent and identically distributed (iid) with probability density function (pdf)

$$f_X(x|\sigma^2) = \left(\frac{1}{\sigma^2}\right)^{\frac{1}{2}} \sqrt{\frac{2}{\pi}} e^{-x^2/2\sigma^2}$$

for $x > 0$ and $\sigma > 0$.

(a) What is the pdf X_1/X_2? Are X_1/X_2 and X_2 independent? Why or why not?

$$\begin{aligned}
U &= X_1/X_2 \text{ and } V = X_2 \quad 1 \\
X_2 &= V \text{ and } X_1 = UV \quad 1 \\
J &= V \quad 1 \\
f_{UV(u,v)} &= f_{X_1X_2}(uv, v)v \\
&= \frac{1}{\sigma^2}\frac{2}{\pi}e^{-(v^2(u^2+1))/2\sigma^2}v
\end{aligned}$$

1 $\quad$ To $\quad$ 1

for $u, v > 0$. $U = X_1/X_2$ and $V = X_2$ are not independent because the joint pdf doesn't factor.

1 $\quad$ 1

$$\begin{aligned}
f_U(u) &= \int_0^\infty \frac{1}{\sigma^2}\frac{2}{\pi}e^{-(v^2(u^2+1))/2\sigma^2}vdv \quad 1 \\
w &= (v^2(u^2+1))/2\sigma^2 \quad 1 \quad 1 \\
dw &= v(u^2+1)/\sigma^2 dv \\
vdv &= \frac{\sigma^2}{u^2+1}dw \\
f_U(u) &= \int_0^\infty \frac{1}{\sigma^2}\frac{2}{\pi}e^{-w}\frac{\sigma^2}{u^2+1}dw \\
&= \int_0^\infty \frac{2}{\pi}e^{-w}\frac{1}{u^2+1}dw \\
&= -\frac{2}{\pi(u^2+1)}e^{-w}\Big|_{w=0}^{w=\infty} \\
&= \frac{2}{\pi(u^2+1)}
\end{aligned}$$

Calc 1

for $u > 0$. Support 1

(b) Let σ^2 have an Inverted-Gamma(α, β) distribution. The pdf is

$$f_{\sigma^2}(\sigma^2) = \frac{1}{\Gamma(\alpha)\beta^\alpha}\left(\frac{1}{\sigma^2}\right)^{\alpha+1} e^{-1/\beta\sigma^2},$$

where $\sigma^2, \alpha, \beta > 0$. What is the distribution of σ^2 given $X_1 = x_1$? What is the distribution of σ^2 given $X_1 = x_1$ and $X_2 = x_2$?

$$
\begin{aligned}
f_{\sigma^2|X_1}(\sigma^2|x_1) &\propto f_{\sigma^2}(\sigma^2) f_X(x_1|\sigma^2) \\
&\propto \left(\frac{1}{\sigma^2}\right)^{\alpha+1} e^{-1/\beta\sigma^2} \left(\frac{1}{\sigma^2}\right)^{\frac{1}{2}} e^{-x_1^2/2\sigma^2} \quad (1) \\
&= \left(\frac{1}{\sigma^2}\right)^{\alpha+\frac{1}{2}+1} e^{-(2/\beta+x_1^2)/2\sigma^2} \quad (1)
\end{aligned}
$$

(1) (1) (1) (1)

Which is the kernel for an Inverted-Gamma$\left(\alpha + \frac{1}{2}, \frac{2}{2/\beta + x_1^2}\right)$ equivalently an Inverted-Gamma$\left(\alpha + \frac{1}{2}, \frac{2\beta}{2+x_1^2\beta}\right)$. (1)

Therefore, the distribution of σ^2 given $X_1 = x_1$ is an Inverted-Gamma$\left(\alpha + \frac{1}{2}, \frac{2\beta}{2+x_1^2\beta}\right)$.

(1) Similarly, the distribution of σ^2 given $X_1 = x_1$ and $X_2 = x_2$ is an Inverted-Gamma$\left(\alpha + 1, \frac{2\beta}{2+(x_1^2+x_2^2)\beta}\right)$.

---

2. Let X have the double exponential distribution with parameters μ and σ:

$$f_X(x|\mu, \sigma) = \frac{1}{2\sigma} e^{-|x-\mu|/\sigma}$$

where $-\infty < x, \mu < \infty$ and $\sigma > 0$. The mean of this distribution is μ and the variance is $2\sigma^2$.

(a) Show that the moment generating function (mgf) of X is

$$M_X(t) = \frac{e^{\mu t}}{1 - (\sigma t)^2}$$

for $|t| < 1/\sigma$. Hint: divide the integral at μ:

$$\int_{-\infty}^{\infty} \cdots dx = \int_{-\infty}^{\mu} \cdots dx + \int_{\mu}^{\infty} \cdots dx$$

where $\cdots$ is something you supply.

$$
\begin{aligned}
M_X(t) &= \int_{-\infty}^{\infty} e^{tx} \frac{1}{2\sigma} e^{-|x-\mu|/\sigma} dx \\
&= \int_{-\infty}^{\infty} \frac{1}{2\sigma} e^{tx-|x-\mu|/\sigma} dx \\
&= \int_{-\infty}^{\mu} \frac{1}{2\sigma} e^{tx+(x-\mu)/\sigma} dx + \int_{\mu}^{\infty} \frac{1}{2\sigma} e^{tx-(x-\mu)/\sigma} dx \\
&= \frac{e^{-\mu/\sigma}}{2\sigma} \int_{-\infty}^{\mu} e^{(t+1/\sigma)x} dx + \frac{e^{\mu/\sigma}}{2\sigma} \int_{\mu}^{\infty} e^{(t-1/\sigma)x} dx \\
&= \frac{e^{-\mu/\sigma}}{2\sigma} e^{(t+1/\sigma)x} \frac{1}{t+1/\sigma}\Bigg|_{-\infty}^{\mu} + \frac{e^{\mu/\sigma}}{2\sigma} e^{(t-1/\sigma)x} \frac{1}{t-1/\sigma}\Bigg|_{\mu}^{\infty} \\
&= \frac{e^{-\mu/\sigma}}{2(t\sigma+1)} e^{(t+1/\sigma)x}\Big|_{-\infty}^{\mu} + \frac{e^{\mu/\sigma}}{2(t\sigma-1)} e^{(t-1/\sigma)x}\Big|_{\mu}^{\infty} \\
&= \frac{e^{-\mu/\sigma}}{2(t\sigma+1)} e^{t\mu+\mu/\sigma} - \frac{e^{\mu/\sigma}}{2(t\sigma-1)} e^{t\mu-\mu/\sigma} \text{ when } |t| < 1/\sigma \\
&= \frac{e^{t\mu}}{2(t\sigma+1)} - \frac{e^{t\mu}}{2(t\sigma-1)} = \frac{e^{t\mu}(t\sigma-1-t\sigma-1)}{2((t\sigma)^2-1)} = \frac{e^{t\mu}}{1-(t\sigma)^2}
\end{aligned}
$$

Calc

(b) Let μ have a Normal(θ, τ^2) distribution and σ^2 have an Inverted-Gamma(α, β) distribution (as in 1b), independent of one another. The mean and variance of the distribution of σ^2 are $[(\alpha-1)\beta]^{-1}$ and $[(\alpha-1)^2(\alpha-2)\beta^2]^{-1}$, respectively. Assuming that the distribution of X given μ and σ is double exponential, what are the (unconditional) mean and variance of X?

$$
\begin{aligned}
E(X) &= E[E(X|\mu,\sigma^2)] = E[\mu] = \theta \\
V(X) &= E[V(X|\mu,\sigma^2)] + V[E(X|\mu,\sigma^2)] = E[2\sigma^2] + V[\mu] \\
&= 2[(\alpha-1)\beta]^{-1} + \tau^2
\end{aligned}
$$

---

3. Let $X_1, \ldots, X_n$ be a random sample form an Inverted-Gamma(α, β) distribution where α is known. Recall from before the pdf of an Inverted-Gamma(α, β) distribution is

$$
f_X(x) = \frac{1}{\Gamma(\alpha)\beta^{\alpha}} \left(\frac{1}{x}\right)^{\alpha+1} e^{-1/\beta x}
$$

for $x, \alpha, \beta > 0$.

(a) Derive the MLE estimator of β.

$$L(\beta) = \prod_{i=1}^{n} \frac{1}{\Gamma(\alpha)\beta^{\alpha}} \left(\frac{1}{x_i}\right)^{\alpha+1} e^{-1/\beta x_i}$$

$$= \left(\frac{1}{\Gamma(\alpha)\beta^{\alpha}}\right)^{n} \left(\frac{1}{\prod_{i=1}^{n} x_i}\right)^{\alpha+1} e^{-(1/\beta)\sum_{i=1}^{n}\frac{1}{x_i}}$$

$$\ell(\beta) = \cdots - n\alpha \ln(\beta) - (1/\beta)\sum_{i=1}^{n}\frac{1}{x_i}$$

$$\frac{\partial \ell}{\partial \beta} = -\frac{n\alpha}{\beta} + (1/\beta^2)\sum_{i=1}^{n}\frac{1}{x_i}$$

$$0 = -\frac{n\alpha}{\hat{\beta}} + (1/\hat{\beta}^2)\sum_{i=1}^{n}\frac{1}{x_i}$$

$$\frac{n\alpha}{\hat{\beta}} = (1/\hat{\beta}^2)\sum_{i=1}^{n}\frac{1}{x_i}$$

$$\hat{\beta} = \frac{\sum_{i=1}^{n}\frac{1}{x_i}}{n\alpha}$$

(b) Derive the 0.05 level LRT test of $H_0 : \beta = \beta_0$ versus $H_1 : \beta > \beta_0$.

If $\hat{\beta} < \beta_0$ then $\frac{L(\hat{\beta}=\beta_0)}{L(\beta_0)} = 1$ and we fail to reject H_o.

Otherwise,

$$
\begin{aligned}
\hat{\beta} &= \frac{\sum_{i=1}^n \frac{1}{x_i}}{n\alpha} \quad 1 \\
\hat{\beta}_0 &= \beta_0 \quad 1 \\
L(\hat{\beta}) &= \left(\frac{1}{\Gamma(\alpha)\hat{\beta}^\alpha}\right)^n \left(\frac{1}{\prod_{i=1}^n x_i}\right)^{\alpha+1} e^{-(1/\hat{\beta})\sum_{i=1}^n \frac{1}{x_i}} \quad 1 \\
L(\beta_0) &= \left(\frac{1}{\Gamma(\alpha)\beta_0^\alpha}\right)^n \left(\frac{1}{\prod_{i=1}^n x_i}\right)^{\alpha+1} e^{-(1/\beta_0)\sum_{i=1}^n \frac{1}{x_i}} \quad 1 \\
\frac{L(\hat{\beta})}{L(\beta_0)} &= \frac{\left(\frac{1}{\hat{\beta}^{n\alpha}}\right) e^{-(1/\hat{\beta})\sum_{i=1}^n \frac{1}{x_i}}}{\left(\frac{1}{\beta_0^{n\alpha}}\right) e^{-(1/\beta_0)\sum_{i=1}^n \frac{1}{x_i}}} > k \quad 1 \\
&\Leftrightarrow \frac{e^{(1/\beta_0)\sum_{i=1}^n \frac{1}{x_i}}}{\hat{\beta}^{n\alpha}} > k_1 \\
&\Leftrightarrow (1/\beta_0)\sum_{i=1}^n \frac{1}{x_i} - n\alpha\ln(\hat{\beta}) > k_2 \quad 1 \\
&\Leftrightarrow \frac{n\alpha\hat{\beta}}{\beta_0} - n\alpha\ln(\hat{\beta}) > k_2 \\
&\Leftrightarrow \hat{\beta} > C \text{ because} \\
&\quad \frac{\partial}{\partial\hat{\beta}}\frac{n\alpha\hat{\beta}}{\beta_0} - n\alpha\ln(\hat{\beta}) = \frac{n\alpha}{\beta_0} - \frac{n\alpha}{\hat{\beta}} > 0 \text{ when } \hat{\beta} > \beta_0 \quad 1
\end{aligned}
$$

The test:

$$
\phi(\boldsymbol{X}) = \begin{cases} 1 & \hat{\beta} > C \\ 0 & \text{otherwise.} \end{cases} \quad 1
$$

Next to find C.

$$
\begin{aligned}
U_i &= \frac{1}{X_i} \quad 1 \\
X_i &= \frac{1}{U_i} \quad 1 \\
J &= -\frac{1}{U_i^2} \quad 1 \\
f_{U_i}(u) &= \frac{1}{\Gamma(\alpha)\beta^\alpha} u^{\alpha+1} e^{-u/\beta} \underbrace{u^{-2}}_{} \quad 1 \\
&= \frac{1}{\Gamma(\alpha)\beta^\alpha} u^{\alpha-1} e^{-u/\beta} \quad 1
\end{aligned}
$$

for $u > 0$. Therefore $U_1 = \frac{1}{X_1}, \ldots, U_n = \frac{1}{X_n}$ are distributed as iid $\underbrace{\text{Gamma}(\alpha, \beta)}_{1}$ random variables.

$$\Rightarrow \sum_{i=1}^{n} \frac{1}{X_i} \sim \text{Gamma}(n\alpha, \beta)$$ 1

$$\Rightarrow \hat{\beta} = \frac{\sum_{i=1}^{n} \frac{1}{X_i}}{n\alpha} \sim \text{Gamma}(n\alpha, \beta/n\alpha)$$

$$\Rightarrow \frac{2n\alpha\hat{\beta}}{\beta} \sim \text{Gamma}(n\alpha, 2)$$

$$\Rightarrow \frac{2n\alpha\hat{\beta}}{\beta} \sim \chi^2_{2n\alpha}$$

1

Under H_0 :

1

$$0.05 = \Pr(\hat{\beta} > C|\beta = \beta_0)$$ 1

$$= \Pr\left(\frac{2n\alpha\hat{\beta}}{\beta_0} > \frac{2n\alpha C}{\beta_0}\middle|\beta = \beta_0\right)$$

1

$$= \Pr\left(\chi^2_{2n\alpha} > \frac{2n\alpha C}{\beta_0}\right)$$

$$\Rightarrow \frac{2n\alpha C}{\beta_0} = \chi^2_{2n\alpha,1-0.05}$$

$$\Rightarrow C = \frac{\beta_0\chi^2_{2n\alpha,1-0.05}}{2n\alpha}$$

1

---

4. A two factor model based on the correlation matrix results in the following factors. Assume the PC method is used to estimate the loadings.

(a) Interpret the meaning of factor 2 (Note: Gaelic=Irish language)

Gaelic	0.4	0.6
English	0.4	0.5
History	0.2	0.5
Arithmetic	0.8	-0.01
Algebra	0.8	-0.1
Geometry	0.6	-0.1

The description of factor 2 should be similar to "language arts plus history" 2

(b) Briefly describe two different ways to assess the fit of the single factor model.

Let ℓ_{ij} = the loading of the i^{th} variable of the j^{th} factor.

- $h_i^2 = \ell_{i1}^2$
- $\psi_i = 1 - \ell_{i1}^2$

4

2 for each way

Score

0 ← missing or wrong

1 ← partial

2 — correct

- $\lambda_1 = \sum_{i=1}^{6} \ell_{i1}^2$
  $\frac{\lambda_1}{6}$ as % explained
- Examine $\boldsymbol{R} - \boldsymbol{L}\boldsymbol{L}' - \boldsymbol{\Psi}$ where

$$\boldsymbol{L} = \begin{pmatrix} \ell_{i1} \\ \ell_{i2} \\ \vdots \\ \ell_{i6} \end{pmatrix}, \quad \boldsymbol{\Psi} = \begin{pmatrix} \psi_1 & & & \\ & \psi_2 & & \\ & & \ddots & \\ & & & \psi_6 \end{pmatrix}$$

  and $\boldsymbol{R}$ is the sample correlation matrix.

---

5. In an experiment, there are six soybean fields. The fields are assigned to different fertilizers. The table below shows the grain yield measured for each of the fields in metric tons/hectare.

Field	Fertilizer	Grain yield in mt/he
1	1	3.98
2	1	3.27
3	1	3.62
4	2	4.00
5	3	4.30
6	3	4.27

Assume the treatments effects model for the described example, where μ represents the overall mean and the different α terms represent the fertilizer effects.

(a) Write out the model in matrix form $\boldsymbol{Y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{e}$.

$$\boldsymbol{Y} = \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \\ Y_5 \\ Y_6 \end{pmatrix} \qquad \boldsymbol{y} = \begin{pmatrix} 3.98 \\ 3.27 \\ 3.62 \\ 4.00 \\ 4.30 \\ 4.27 \end{pmatrix}$$ 1

$$\boldsymbol{X} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$ 1

$$\boldsymbol{\beta} = \begin{pmatrix} \mu \\ \alpha_1 \\ \alpha_2 \\ \alpha_2 \end{pmatrix}$$ 1

$$\boldsymbol{e} = \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \end{pmatrix}$$ 1

$$\boldsymbol{e} \sim \mathrm{N}(\boldsymbol{0}, \boldsymbol{I}\sigma^2)$$ 2

(b) Derive the projection matrix. Show your work.

$$
\boldsymbol{X}'\boldsymbol{X} = \begin{pmatrix} 6 & 3 & 1 & 2 \\ 3 & 3 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 2 & 0 & 0 & 2 \end{pmatrix}
$$ 1

$$
\left(\boldsymbol{X}'\boldsymbol{X}\right)^{-} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1/3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1/2 \end{pmatrix}
$$ 1

$$
\left(\boldsymbol{X}'\boldsymbol{X}\right)^{-}\boldsymbol{X}' = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1/3 & 1/3 & 1/3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/2 & 1/2 \end{pmatrix}
$$ 1

$$
\boldsymbol{X}\left(\boldsymbol{X}'\boldsymbol{X}\right)^{-}\boldsymbol{X}' = \begin{pmatrix} 1/3 & 1/3 & 1/3 & 0 & 0 & 0 \\ 1/3 & 1/3 & 1/3 & 0 & 0 & 0 \\ 1/3 & 1/3 & 1/3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 0 & 0 & 1/2 & 1/2 \end{pmatrix}
$$ 1

(c) Which of the linear combinations of estimators are estimable, when there are no restrictions of any of the parameters? Explain. In the case that a combination is estimable, interpret its meaning (in plain English words).

i. μ 1

Not estimable. Because the expected value of each observation contains both and μ and α_i any linear combination of observations would have an expected value that contains μ would of necessity also contain at least one $b_i\alpha_i$ where $b_i \neq 0$. 1

ii. $\mu + \alpha_2$

1 Estimable, average yield of fertilizer 2. 1
The expected value of y_4 is equal to $\mu + \alpha_2$ 1

iii. α_1 1

Not estimable. Because, every observation that contains α_1 also contains μ, therefore any linear combination of observations whose expectation contains α_1 also contains μ. 1

iv. $\alpha_2 - \alpha_1$

1

Estimable, difference between in average yield between fertilizers 2 and 1.

1

The expected value of $y_4 - y_1$ is equal to $\alpha_2 - \alpha_1$ 1

(d) Pick a set of estimable functions, which readily allows the comparison of the fertilizers, and write out the corresponding full rank matrix $\boldsymbol{X}^*$ together with the new parameter vector $\boldsymbol{\beta}^*$.

One set.

3

- $\beta_1^* = \mu + \alpha_1$ average yield of fertilizer 1.
- $\beta_2^* = \alpha_2 - \alpha_1$ average additional yield when using fertilizer 2 compared to fertilizer 1.
- $\beta_3^* = \alpha_3 - \alpha_1$ average additional yield when using fertilizer 3 compared to fertilizer 1.

$$\boldsymbol{X}^* = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$ 1

$$\boldsymbol{\beta}^* = \begin{pmatrix} \beta_1^* \\ \beta_2^* \\ \beta_3^* \end{pmatrix}$$ 1

(e) State and explain properties of a BLUE estimator of an element of $\boldsymbol{\beta}^*$.
Sketch a proof of the properties.

- It is an unbiased estimator of β_1^*. That is on average the estimator is equal to β_1^* 1
- It is a linear function of Y_i's. That is $\beta_1^* = a + \sum_{i=1}^{6} b_i Y_i$. Unbiased $\Rightarrow a = 0$ 1
- Among linear and unbiased estimators of β_i^* it has the smallest variance. 1

Outline of a proof:

linear and unbiased:

$$\hat{\beta}_1^* = \boldsymbol{k}'\boldsymbol{\beta}^*$$

1 $$E(\boldsymbol{a}'\boldsymbol{y}) = \boldsymbol{k}'\boldsymbol{\beta}^* = \beta_1^*$$ 1

$$\Rightarrow \boldsymbol{a}'\boldsymbol{X}^* = \boldsymbol{k}' = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$ 1

minimum variance

$$f(\boldsymbol{a}) = \boldsymbol{a}'\boldsymbol{a} + 2(\boldsymbol{a}'\boldsymbol{X}^* - \boldsymbol{k}')\boldsymbol{\lambda}$$ 1

partials with regard to $\boldsymbol{a}$ and $\boldsymbol{\lambda}$ 1

Set derivatives equal to zero and solve

1

(f) In a larger multi-state experiment, a larger number of soybean fields was evaluated for 6 years. For fertilizer 1 25 fields were evaluated, and 27 fields were evaluated for both fertilizer 2 and 3. We want to compare two models:

i. Model A: $y_{itj} = \mu + \alpha_i + \beta t + e_{itj}$

ii. Model B: $y_{itj} = \mu + \alpha_i + \beta_i t + e_{itj}$

with μ the expected average grain yield, α_i are the fertilizer-specific effects ($i = 1, 2, 3$; $t = 1, 2, 3, 4, 5, 6$; $j = 1, \ldots, 25, 26, 27$, respectively). We assume that the grain yield is normally distributed.

i. Draw a sketch of expected grain yield versus time for each model. Labels (1)

For model A: The plot will consist of 6 parallel lines, one for each fertilizer
For model B: The plot will consist of 6 lines, one for each fertilizer. It should be clear that the lines do not need to be parallel.

ii. Explain the difference between the models.

Model A assumes that the average yield of every fertilizer increases/decreases by the same amount each year.
Model B says that average yield of each fertilizer increases/decreases by the same amount each year, but that amount can vary between fertilizers.