Stat 882

1. A box contains 10 fuses. Eight of them are rated 10 amperes (A) and the other two are rated 15 A. Fuses are selected at random until a 15 A fuse is selected.

a) What is the probability that at least three fuses are selected from the box?

b) What is the probability that exactly four fuses are selected given that at least three fuses are selected?

2. Let X be a random variable with pdf

$$f_X(x) = \frac{3}{16}x^2; \quad -2 < x < 2.$$

a) Verify that $f_X(x)$ is a pdf.

b) Find the $E[X^c]$ where *c* is a positive integer.

3. The speed of a molecule of gas at equilibrium is a random variable S with pdf

$$f_S(s) = as^2 e^{-bs^2}; \quad s > 0$$

where *a* is the normalizing constant and *b* depends on the temperature of the gas and the mass of the molecule. Find the pdf of the kinetic energy $K = \frac{mS^2}{2}$ of the molecule, where *m* is the mass of the molecule (a constant).

4. Let X and Y be random variables with bivariat pdf

$$p_{XY}(x, y) = ye^{-xy}; \quad x > 0, \ 0 < y < 1.$$

a) Find the pdf of Y.

b) Are X and Y independent random variables? Why or why not?

c) Find the Pr(X > 1/Y).

5. Let X be a random variable with pmf

$$f_X(x) = \binom{n}{x} p^x (1-p)^{n-x}; \quad x = 0, 1, \ldots, n; \quad 0 < p < 1$$

Find the E[X]. Do not use the moment generating function approach.

Stat 882

1) Let X be distributed as a beta random variable with pdf

$$f_X(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(beta)} x^{\alpha-1}(1-x)^{\beta-1}, \quad 0<x<1, \quad \alpha>0, \quad \beta>0.$$

a) Derive the mean of X.

b) Show that the beta family of distributions is a member of the exponential family.

2) Let X be distributed as binomial random variable with pmf

$$f_X(x|n,p) = \binom{n}{x} p^x (1-p)^{n-x}, \quad x=0,1,\ldots,n \quad n=1,2,\ldots, \quad 0<p<1.$$

a) Derive the mean of X without using the mgf of X.

b) Find the moment generating function of X.

3) Let X and Y be continuous random variables having joint density f given by

$$f_{X,Y}(x,y) = 3\text{xy}, \quad 0<x<1, \quad x<y<2-x.$$

a) Find Pr(Y < 1).

b) Find the marginal density of X.

c) Find the conditional pdf of Y|X.

d) Find $E[Y|X=x]$.

4) (Chebychev's Inequality) Let X be a random variable and let $g(x)$ be a nonnegative function, then for any $r > 0$,

$$Pr(g(X) \geq r) \leq \frac{E[g(X)]}{r}.$$

Prove Chebychev's Inequality for the case when X is a continuous random variable.

Stat 882

1. Suppose that X and Y are jointly distributed random variables with the joint pdf given by

$$f_{X,Y}(x,y) = 4x\ ; 0 < x < 1\ ; 0 < y < x^2$$

a) Find the marginal pdf of X.
b) Find the marginal pdf of Y.
c) Find the E[XY].
d) Find $f_{X|Y}(x|y)$.

2. Let x and Y be distributed as bivariate normal random variables with pdf

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left\{ \frac{-1}{2(1-\rho^2)}(x^2 - 2\rho xy + y^2) \right\}$$

Show that the conditional distribution of X given Y is $N(\rho y, 1-\rho^2)$.
Note: you may assume that the marginal distributions of both X and Y are $N(0,1)$.

3. Let $X_1, X_2, \ldots, X_5$ and $U_1, U_2, \ldots, U_5$ be independent random variables where $X_i \sim N(4,8)$ and $U_i \sim gamma(4,4)$. Give the distribution (including parameters) of each of the following:

a) Explain why $U_1/2$ is a chi-squared(*p*) random variable. What is *p*?
b) $X_1 + 3X_2$
c) $(U_1 + U_2)/(U_3 + U_4)$
d) $(X_1 - X_2)/\sqrt{U_3}$

4) Let X and Y be independent random variables with X~N(0, 1) and Y~gamma(α, β).

a) Find the joint pdf of $U = X/\sqrt{Y}$ and $V = Y$.
b) Find the pdf of U.

5) Show that the moment generating function of Poisson(λ) is

$$M_X(t) = e^{\lambda(e^t - 1)}$$

6) Let $X_1, \ldots, X_n$ be a random sample from Poisson(λ).

a) Derive the distribution of $T = \sum_{i=1}^{n} X_i$
b) Derive the distribution of $T | X_1 = 0$. Recall that $T = X_1 + \sum_{i=2}^{n} X_i$.

Stat 883

1. Let $X_1, \ldots, X_n$ be a random sample from a population with

$$f_X(x|\alpha) = \frac{3\alpha^3}{x^4}; \alpha < x; \alpha > 0,$$
$$E[X] = \frac{3\alpha}{2}$$
$$Var[X] = \frac{3\alpha^2}{4} \quad .$$

a) Find the method of moments estimator of α .
b) Find the MLE of α . Be sure to verify that it is in fact the MLE of α .
c) Find the bias, standard error, and mean squared error of the method of moments estimator of α .
d) Find the bias, standard error, and mean squared error of the MLE of α .

2) Let $X_1, \ldots, X_n$ be a random sample from a $normal(0, \sigma^2)$ distribution.
a) Find the method of moments estimator of σ^2 .
b) Find the MLE estimator of σ^2 .
c) Find the MLE estimator of $\sigma = \sqrt{\sigma^2}$.
d) Find the bias, standard error, and mean squared error of the MLE of $\sigma = \sqrt{\sigma^2}$.

Stat 883

1. Let $X_{1,\ldots,}X_n$ be iid random variables with pdf $f_X(x|\theta)=e^{-x+\theta}$ for $x\geq\theta$ and with cdf $F_X(x)=1-e^{-x+\theta}$ for $x\geq\theta$.

a) Derive the MLE of θ .

b) Drive the LRT of size α of $H_o:\theta=\theta_o$ versus $H_a:\theta>\theta_o$.

c) Derive the power function for this test.

2. Consider a random sample $X_{1,\ldots,}X_n$ from a $uniform(\theta_1,\theta_2)$ distribution. The pdfs of the smallest and largest order statistics are,

$$f_{X_{(1)}}(x)=\begin{cases}\dfrac{n(\theta_2-x)^{n-1}}{(\theta_2-\theta_1)^n} & \theta_1<x<\theta_2\\ 0 & elsewhere\end{cases} \quad \text{and} \quad f_{X_{(n)}}(x)=\begin{cases}\dfrac{n(x-\theta_1)^{n-1}}{(\theta_2-\theta_1)^n} & \theta_1<x<\theta_2\\ 0 & elsewhere\end{cases}$$

respectively. The expected values of the smallest and largest order statistics are,

$$E[X_{(1)}]=\theta_1+\frac{(\theta_2-\theta_1)}{n+1} \quad \text{and} \quad E[X_{(n)}]=\theta_2-\frac{(\theta_2-\theta_1)}{n+1} \quad \text{respectively.}$$

a) Prove that $X_{(1)}$ and $X_{(n)}$ are jointly sufficient for θ_1 and θ_2 .

b) Show that $X_{(n)}$ is complete for θ_2 .

c) It turns out that $X_{(1)}$ and $X_{(n)}$ are also jointly complete for θ_1 and θ_2 . We are interested in estimating the mean of the distribution: $\mu=\dfrac{\theta_1+\theta_2}{2}$. Find the UMVUE for μ .

3. Rao-Blackwell Theorem

Let W be any unbiased estimator of $\tau(\theta)$, and let T be a sufficient statistic for θ . Define $\phi(T)=E(W|T)$. Then $E_\theta[\phi(T)]=\tau(\theta)$ and $Var_\theta[\phi(T)]\leq Var_\theta(W)$.

a) Show that $E_\theta[\phi(T)]=\tau(\theta)$.

b) Show that $Var_\theta[\phi(T)]\leq Var_\theta(W)$.

c) Explain why the theorem requires T to be a sufficient statistic.

Stat 883

1. Suppose that $X_1, \dots, X_n$ be a random sample from a Poisson(λ) distribution.

a) Derive the MLE of λ.

b) Show that $\phi(x) = \begin{cases} 1 & \sum_{i=1}^{n} X_i > K \\ 0 & \sum_{i=1}^{n} X_i \le K \end{cases}$ is a level α UMP test for $H_0: \lambda \le 1$ versus $H_1: \lambda > 1$.

c) Find the size of $\phi(X)$. The answer may be left in terms of summation notation.

d) Find the UMVUE of $\lambda e^{-\lambda}$.

2. Let X be a random sample of size 1 of a discrete random variable with pmf

$$f(x|\theta) = \frac{x-\theta}{10}; x = \theta+1, \dots, \theta+4$$

Note: $E(X) = \theta + 3$ and $Var(X) = 1$.

a) Find the MLE of θ.

b) Show that $Y = X - \theta$ is a pivotal quantity for θ.

c) Show that $X - 4 \le \theta \le X - 2$ is a confidence interval and find its confidence coefficient.

d) Use the confidence interval to construct a test of $H_0: \theta = 0$ versus $H_1: \theta \ne 0$.

e) Find the power function of the test at θ = -2, 2 and 2.5.

3. The proof of the Cramér-Rao inequality theorem includes the following equality,

$$Var_\theta\left(\frac{dl(\theta)}{d\theta}\right) = E_\theta\left[\left(\frac{dl(\theta)}{d\theta}\right)^2\right].$$

Let $X_1, \dots, X_n$ be a random sample from a uniform(0, θ) where

$$E(X_i) = \frac{\theta}{2} \text{ and } Var(X_i) = \frac{\theta^2}{12}.$$

a) Show that the equality doesn't hold for this distribution.

b) What condition of the Cramér-Rao inequality wasn't satisfied and why wasn't it satisfied in this case?

Stat 970

1) State Theorem 7.

2) Prove that $X(X'X)^{-} X' = X$

3) Prove that rank X is equal to the rank X'X.

4) Let $y \sim N(X\beta, I\sigma^2)$ and $\hat{\beta} = (X'X)^{-} X'y$. Find the distribution with parameters of $U = y'X\hat{\beta}$.

5) Let $y_{ij} = \mu + a_i + b_j + e_{ijk}$ where y_{ij} is the dependent variable, μ is the intercept, a_i is the effect of level *i* of treatment factor A, b_j is the effect of level *j* of treatment factor B, and e_{ijk} is the residual. The data are summarized in the following table:

Y	A	B
1	1	Red
2	1	Red
3	1	Green
1	2	Red
1	2	Green
2	2	Green

Set up the normal equations.

6) Let $y \sim N(1\mu, I\sigma^2 + J\tau^2)$ where *y* is a *n* x 1 random vector. Recall that

$$J = 11' \text{ and } C = I - \bar{J} = I - \frac{1}{n}J.$$

a) Find the distribution with parameters of $\bar{y} = \frac{1}{n}1'y$.

b) Find the distribution with parameters of $y'Cy$

c) Find the distribution with parameters of $\frac{\bar{y}}{\sqrt{y'Cy}}$.

Stat 970

1) Let $y \sim N(X\beta, I\sigma^2)$ and $k'\beta$ be an estimable function. Derive the Best Linear Unbiased Estimator of $k'\beta$.

2. Let $y \sim N(X\beta, I\sigma^2)$ and $k'\beta$ be a scalar estimable function.

a) What are distributions, with parameters, of $k'\hat{\beta}$ and SSE? Note: I'm not asking for you to derive the distributions of $k'\hat{\beta}$ and SSE.

b) Let *w* be a normal random variable with mean $k'\beta+\tau$ and variance σ^2. In addition, *w* is distributed independently of y. Find the distribution of $w-k'\hat{\beta}$.

c) Construct a t-statistic for testing the hypothesis that $\tau=0$.

3) Let $y_{ijk}=\mu+\alpha_i+\beta_j+\gamma_{ij}+e_{ijk}$ where *i* = 1, 2, *j* = 1, 2, *k* = 1, …, *n*, and e_{ijk} are iid $N(0,\sigma^2)$.

a) Set up and solve the normal equations.

b) Show which of the following are or are not estimable.

i) $\alpha_1-\alpha_2$

ii) $\mu+\alpha_1+\beta_2+\gamma_{12}$

iii) $\beta_1-\beta_2+\gamma_{11}+\gamma_{12}$

c) Construct the SSH for testing

$$H_o: \mu+\alpha_1+\beta_1+\gamma_{11}=\mu+\alpha_2+\beta_2+\gamma_{22}$$
$$H_1: \mu+\alpha_1+\beta_1+\gamma_{11}\neq\mu+\alpha_2+\beta_2+\gamma_{22}.$$

Be sure to simplify your result.

Appendix

$$\frac{dx}{dx'}=\frac{dx'}{dx}=I$$
$$\frac{dAx}{dx'}=A$$
$$\frac{dx'B}{dx}=B$$
$$\frac{dx'Ax}{dx}=2\text{Ax}$$
$$\frac{dx'Ax}{dx'}=2\text{x}'A$$
$$\frac{du'v}{dx'}=u'\frac{dv}{dx'}+v'\frac{du}{dx'}$$
$$\frac{d\,tr[H]}{dy}=tr\left(\frac{dH}{dy}\right)$$
$$\frac{dV^{-1}}{dy}=-V^{-1}\frac{dV}{dy}V^{-1}$$
$$\frac{d\ln(|V|)}{dy}=tr\left(V^{-1}\frac{dV}{dy}\right)$$

Stat 970

1. Let $y \sim N(X\beta + W\theta, I\sigma^2)$ where $\hat{\beta} = (X'X)^{-} X'y$.

a) Derive the distribution of $\hat{y}_0 = X\hat{\beta}$.

b) Derive the distribution of $SSO = \hat{y}_0{}'\hat{y}_0$.

c) Derive the distribution of SSO/SSE. Note: SSE is based on the full design matrix (X W). You do not need to show that $SSE/\sigma^2 \sim X^2_{N-p^*}$ where p^* is the rank of the full design matrix.

2. Let $y_i = a + b(X_i - \bar{x}) + e_i$ where i = 1, …, N and the e_i are iid $N(0, \sigma^2)$. Assume that

$$s_x^2 = \frac{\sum^N (x_i - \bar{x})^2}{N}$$

is greater than 0.

a) Set up and solve the normal equations.

b) Show that *a* and *b* are estimable.

c) Recall that the leverage of observation *i* is the i^{th} diagonal element of the Hat matrix. Find the leverage of an observation when:

i) $x_i - \bar{x} = 0$

ii) $x_i - \bar{x} = s_x$.

3. Let $y_{ijk} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + e_{ijk}$ where the e_{ijk} are iid $N(0, \sigma^2)$. Using the data given below answer the following questions.

	B		
A	1	2	3
1	10, 12	21	22, 24
2	77	23	12, 13, 14
3	10	20	20, 40

Recall that a solution for $\hat{\beta}$ is

$$\hat{\mu} = \hat{\alpha}_i = \beta_i = 0 \quad \text{and} \quad \widehat{(\alpha\beta)}_{ij} = \bar{y}_{ij.}.$$

a) Find the **estimate** of the least square mean for the third level of factor A.

b) Find the distribution with parameters of the least square mean **estimator** for the third level of factor A.

c) What are the hypotheses being tested by the Type I and Type III SS's for A and A*B?

d) Show that the BLUE of

$$\mu + \alpha_1 + \frac{2\beta_1 + \beta_2 + 2\beta_3}{5} + \frac{2(\alpha\beta)_{11} + (\alpha\beta)_{12} + 2(\alpha\beta)_{13}}{5}$$

is the sample mean for the first level of A.

Stat 883

1. Seventy percent of the light aircraft that disappear while in flight are subsequently discovered. Of the aircraft discovered, 60% have an emergency locator, whereas 90% of the aircraft not discovered do not have such a locator. Suppose that a light aircraft has disappeared. If it has an emergency locator, what is the probability that it will be discovered?

2. Let A, B, and C be events defined on a sample space S. Show that $(A \cup B)^C = A^C \cap B^C$. Note: A Venn diagram is not a proof.

3. A computer password consists of eight characters.

a) How many different passwords are possible if each character may be any lowercase letter?

b) A computer system requires that the first three characters are not the same. If eight characters are generated at random, what is the probability that a valid password will be generated?

c) Another computer system instead requires that all the characters are different. If eight characters are generated at random, what is the probability that a valid password will be generated?

4. Let X be a random variable with pdf

$$f_X(x) = \frac{(\alpha+\beta+1)!}{\alpha!\,\beta!} x^{\alpha}(1-x)^{\beta}, \quad 0 < x < 1; \quad \alpha > 0; \quad \beta > 0.$$

Derive the mean and variance of X.

5. Let X be a random variable with pdf

$$f_X(x) = 12[x^2 - x^3], \quad 0 < x < 1.$$

Let $Y = 1 - 1/X$. Find the distribution of Y.

6. Let X be a random variable with pdf

$$f_X(x) = 9x e^{-3x}, \quad 0 < x.$$

a) Verify that this is a pdf.

b) Find the cumulative distribution function (CDF) of X.

Stat 882

1. Let X be distributed as a normal random variable with mean 0 and σ^2.

   a) Show that the moment generating function of X is

$$M_X(t)=e^{\frac{t^2\sigma^2}{2}}.$$

   b) Derive the mean and variance of X.

2. Let X and Y be continuous random variables having joint density *f* given by

$$f_{XY}(x,y)=\{y e^{y-x} \quad 0<y,\ \ x>2y.$$

   a) Find the marginal density of Y.

   b) Find the conditional pdf of X|Y.

   c) Find the $E(XY^2|Y=y)$.

3. Let X and Y be discrete random variables where

$$f_{XY}(x,y)=\binom{y}{x}\frac{p^x(1-p)^{y-x}}{n};\ \ x=0,1,\ldots,y;\ \ y=1,2,\ldots,n$$

   a) Find the marginal distribution of Y.

   b) Find the $E(X|Y=y)$.

   c) Find the $E(X)$.

4. Let X have a pdf with the form

$$f(x|\theta)=h(x)c(\eta)\exp[t(x)\eta].$$

Start with the equality

$$\int_{-\infty}^{\infty} h(x)c(\eta)\exp[t(x)\eta]dx=1,$$

differentiate both sides, and then rearrange the terms to show that $E[t(X)]=\dfrac{c'(\eta)}{c(\eta)}$.

Stat 882

1. Let X be distributed as a gamma random variable with parameters α and β. Recall that the pdf of the gamma(α, β) is

$$f_X(x) = C x^{\alpha-1} e^{-x/\beta}; \quad x \geq 0.$$

   a) Show that the normalizing constant C is equal to

$$\frac{1}{\Gamma(\alpha)\beta^{\alpha}}.$$

   b) Derive the moment generating function of X.
   c) Derive the mean and variance of X.

2. Let X and Y be continuous random variables having joint density f given by
$f(x, y) = x e^{-y}, \quad 0 \leq x \leq y.$
   a) Find the marginal density of X.
   b) Find the marginal density of Y.
   c) Find the $E(XY)$.
   d) Find the conditional pdf of $f_{X|Y}(x|y)$ and $f_{Y|X}(y|x)$.
   e) Find the $E(Y|X)$.

3. Let X and Y be independent normal random variables, $X \sim Normal(\mu_X, \sigma_X^2), Y \sim Normal(\mu_Y, \sigma_Y^2)$, and let $W = X - Y$.

$$f(x|\mu, \sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/(2\sigma^2)}, \quad -\infty < x < \infty, \quad -\infty < \mu < \infty, \quad \sigma > 0$$

$$E(X) = \mu \quad Var(X) = \sigma^2$$

$$M_X(t) = e^{\mu t + \sigma^2 t^2/2}$$

Derive the distribution of W.

4. Show that if X and Y are independent continuous random variables with pdf $f_X(x)$ and $f_Y(y)$, then $E[g(X)h(Y)] = E[g(X)]E[h(Y)]$.

Stat 882

1. Let $Z \sim N(0,1)$. Derive the distribution of $U = Z^2$.

2. Suppose X and Y are jointly distributed random variables with joint pdf given by

$$f_{XY}(x,y) = \begin{cases} 2xe^{-y} & 0<x<1,\ 0<y \\ 0 & else \end{cases}$$

Note that $E(X) = \frac{2}{3}, E(X^2) = \frac{1}{2}, E(Y) = 1, E(Y^2) = 2.$

a) Find the joint pdf of U = 2X+Y and V = Y.
b) Find E(U) and Var(U).
c) Find E(V) and Var(V).
d) Find E(UV).
e) Find the covariance between U and V.
f) Find the correlation between U and V.
g) Find $f_{U|V}(u|v)$.
h) Find E(U|V).

3. Let $X_1, X_2, \ldots, X_n$ be a random sample from $X_i \sim gamma(\alpha, 2)$.

a) Find the moment generating function of X_1.
b) Using the moment generating function, find the mean and variance of X_1.
c) Find the distribution of $Y_n = \sum_{i=1}^{n} X_i$.
d) The distribution of Y_n is known by two different names. What are the two names and associated parameters?

4. Suppose that $X_i \sim N(7,9), i=1,\ldots,n$ and $Z_j \sim N(0,1), j=1,\ldots,k$, and all variables are independent. Give the distribution (including parameters) of each of the following:

a) $\dfrac{Z_1^2 + Z_2^2}{Z_3^2 + Z_4^2}$

b) $\dfrac{Z_1 - Z_2}{\sqrt{Z_3^2 + Z_4^2}}$

c) $\dfrac{(Z_1 - Z_2)^2}{2} + \dfrac{(Z_1 - 2Z_2 + Z_3)^2}{6}$

d) $2Z_1^2$

e) The joint distribution of $X_1 + 2X_2$ and $2X_1 + X_2$.

5. Suppose we have a box with three red balls, four blue balls, and six green balls. If you draw at random 3 balls without replacement from the box, what is the probability that the balls will all be the same color?

Stat 882

1. Santa Claus is addicted to chocolate-covered caramels. Because of this, he steals one piece of chocolate at random from each of the gift boxes he delivers for Christmas. Half (1/2) of the boxes do not contain any chocolate-covered caramels at all. In one third (1/3) of the boxes, one piece in twenty (1/20) is a chocolate-covered caramel. In the rest of the boxes half of the pieces are chocolate-covered caramels.

a) Calculate the probability that a randomly chosen chocolate piece from a randomly chosen box is a caramel.

b) Calculate the probability that half of the pieces in the chosen box are chocolate-covered caramels, given that the stolen piece was a caramel.

2. Show that the moment generating function of $N(\mu,\sigma^2)$ is

$$M_X(t)=\exp\left(t\mu+\frac{t^2\sigma^2}{2}\right).$$

3. Let $X_1,\ldots,X_n$ be a random sample from $N(\mu,\sigma^2)$. Derive the distribution of $\bar{X}$.

4. Let $X_i, i=1,2,3,4,5$ be independent random variables with $N(\mu,16)$ distributions. Use the $X_i s$ to construct a statistic with the indicated distribution.

a) chi squared with 2 degrees of freedom.

b) normal with mean 0 and variance 1.

c) t distribution with 2 degrees of freedom.

5. Let X and Y be independent random variables with $X \sim gamma(\alpha,1)$ and $Y \sim gamma(\beta,1)$.

a) Find the joint pdf of U = X and V = X/(X+Y).

b) Find the marginal distribution of V.

c) Find $f_{U|V}(u|v)$.

d) Find E(U|V).

e) Find E(U) using part (d). If you prefer, you may answer this question using $\alpha=4$ and $\beta=3$.

6. Let $X_1,\ldots,X_n$ be a random sample where $\bar{X}$ and S^2 are calculated in the usual way. Show that

$$S^2=\frac{1}{2n(n-1)}\sum_{i=1}^{n}\sum_{j=1}^{n}(X_i-X_j)^2.$$

Stat 882

1. Let X be a random variable with pmf

$$f_X(x)=\begin{cases}\dfrac{\lambda^x}{(e^{\lambda}-1)x!} & x=1,2,\ldots\\ 0 & otherwise\end{cases}$$

a) Derive the moment generating function of X.

b) Using either the moment generating function or cumulant generating function, derive the mean and variance of X.

2. Suppose X and Y are jointly distributed random variables with joint pdf given by

$$f_{XY}(x,y)=\begin{cases}6xy^2 & 0\le x\le 1,\ \ 0\le y\le 1\\ 0 & else\end{cases}$$

Note that $E(X)=\frac{2}{3}, E(X^2)=\frac{1}{2}, E(Y)=\frac{3}{4}, E(Y^2)=\frac{3}{5}$.

a) Find the joint pdf of U = X + Y and V = X.

b) Find E(U) and Var(U).

c) Find E(V) and Var(V).

d) Find E(UV).

e) Find the covariance between U and V.

f) Find the correlation between U and V.

g) Find $f_{U|V}(u|v)$.

h) Find E(U|V).

3. Let $X_1, X_2, \ldots$ be a sequence of random variables with $X_n \sim binomial(n, \mu/n)$ with $0<\mu<1$. Show that as $n\to\infty, X_n$ converges to a $Poisson(\mu)$ in distribution.

4. Let $X_1, X_2, \ldots, X_n$ be independent random variables with $X_i \sim gamma(\alpha_i, \beta)$. Derive the distribution of $\bar{X}=\sum_{i=1}^{n} X_i/n.$

5. Suppose that $X_i \sim N(\mu,\sigma^2), i=1,\ldots,n$ and $Z_j \sim N(0,1), j=1,\ldots,k,$ and all variables are independent. Give the distribution (including parameters) of each of the following.

a) $X_1-2X_2+3X_3$

b) $\sum_{i=1}^{n} Z_i^2$

c) $\dfrac{(Z_1-Z_2)^2}{2}+\dfrac{(Z_1+Z_2)^2}{2}$

d) $\dfrac{Z_1^2}{Z_2^2}$

e) $\dfrac{\sum_{i=1}^{n}(X_i-\mu)}{\sqrt{n\sigma^2\sum_{j=1}^{2}(Z_j-\bar{Z})^2}}$

Stat 883

1. Suppose that $X_{1,} \ldots, X_n$ are iid random variables from geometric(p) distribution. Recall that for a geometric(p) distribution:

$$Pr(X=x|p)=p(1-p)^{x-1}; \;\; x=1,2,\ldots; \;\; 0\le p\le 1,$$
$$E(X)=\frac{1}{p}$$
$$var(X)=\frac{1-p}{p^2}.$$

a) Find a complete sufficient statistic for p.
b) Find the MLE of p.
c) Find the MME of p.
d) Suppose that the prior distribution of p is beta(α, β). Find a Bayes estimator of p.

2. Let X be a random variable with a cdf of

$$F_X(x|\theta)=\begin{cases}\frac{X^3}{\theta^3} & 0\le x\le\theta\\ 0 & otherwise\end{cases}$$

where is θ unknown. Let $X_{1,} \ldots, X_n$ be a random sample from this distribution.
a) Show that $X_{(n)}$ is a sufficient statistic.
b) Is $X_{(n)}$ a minimal sufficient statistic? Prove or disprove.
c) Find the pdf and cdf of $X_{(n)}$.
d) Find the MME of θ.
e) Find the MLE of θ.

Stat 883

1. Let $X \sim Binomial(n, p)$. One observation on X is available, and it is known that n is either 2 or 3 and p = 1/2 or 1/3. Based on the following table of distribution, derive the maximum likelihood estimator of the pair (n, p).

	(n, p)			
X = x	(2, 1/2)	(2, 1/3)	(3, 1/2)	(3, 1/3)
0	1/ 4	4/ 9	1/ 8	8/27
1	1/ 2	4/ 9	3/ 8	4/ 9
2	1/ 4	1/ 9	3/ 8	2/ 9
3	0	0	1/ 8	1/27

2. Let $X_{1,} \ldots, X_n$ iid $Poisson(\lambda_1), Y_{1,} \ldots, Y_n$ iid $Poisson(\lambda_2)$ and X_i and Y_j be independent for all i, j. Here $\lambda_1 > 0$ and $\lambda_2 > 0$ are unknown parameters.

a) Find the MLE of λ_1 and λ_2.

b) Identify a complete and sufficient statistic for (λ_1, λ_2).

c) Let (λ_1, λ_2) have a prior distribution with joint pdf

$$\pi(\lambda_{1,} \lambda_2) = \exp[-\lambda_1 - \lambda_2], \quad \lambda_1 > 0, \quad \lambda_2 > 0.$$

Find the posterior distribution of (λ_1, λ_2) and compute the Bayes estimator of λ_1.

3. Let $X_{1,} \ldots, X_n$ iid $N(\mu, 1)$.

a) Identify a sufficient and complete statistic for μ.

b) Show that $X_1 - \bar{X}$ is an ancillary statistic, where $\bar{X} = n^{-1} \sum_{i=1}^{n} X_i$.

c) Using the answer of (a) and (b), argue that $\bar{X}$ and $X_1 - \bar{X}$ are independent.

d) Let $\tau(\mu) = P(X_1 \leq 2)$. Derive the MLE of $\tau(\mu)$.

4. Let $X_{1,} \ldots, X_n$ iid $f(x|\beta, \tau)$, where

$$f(x|\beta, \tau) = \frac{1}{\beta} \exp\left[-\left(\frac{x-\tau}{\beta}\right)\right], \quad x \geq \tau,$$

for $\beta > 0$ and $-\infty < \tau < \infty$. [If X has this distribution, then $X = Y + \tau$ where Y has the usual exponential distribution with mean β.]

a) Identify a two-dimensional sufficient statistic for the parameter (β, τ) and carefully argue that it is sufficient.

b) Find a MME of (β, τ) using the population and sample mean and standard deviation. That is, solving the linear system $E(X_1) = \bar{X}, \sqrt{Var(X_1)} = \sqrt{S^2}$.

c) Identify the MLE of (β, τ) and carefully argue that it does indeed maximize the likelihood.

Stat 883

1. Suppose that $X_{1,}\ldots,X_n$ are iid random variables from a discrete distribution with the following properties.

$$Pr(X=x|\theta)=\frac{\theta^x}{(\theta+1)^{x+1}};\quad x=0,1,\ldots;\quad \theta>0,$$
$$E(X)=\theta$$
$$var(X)=\theta(\theta+1)$$
$$Pr(\sum_{i=1}^{n}X_i=t|\theta)=\binom{n+t-1}{t}\frac{\theta^t}{(\theta+1)^{t+n}}.$$

a) Derive the MLE of θ.
b) **Derive** the CRLB for unbiased estimators of θ.
c) Find the UMVUE for θ. Show that it is the UMVUE
d) Find the CRLB for unbiased estimators of $1/(\theta+1)$.
e) Find the UMVUE for $1/(\theta+1)$. Show that it is the UMVUE.

2. Suppose that $X_{1,}\ldots,X_n$ is a random sample from a continuous distribution with pdf

$$f(x;\theta)=\theta x^{\theta-1};\quad 0\le x\le 1,\quad \theta>0$$

a) Derive the most powerful size-α test for testing $H_0:\theta=\theta_0$ versus $H_1:\theta=\theta_1$ where $\theta_0>\theta_1$.
b) Find the critical region when $n=1,\theta_0=.5,\theta_1=.25,$ and $\alpha=.05$.

3. Suppose we have a statistic *T* which is distributed exp(λ) and a test

$$\phi(T)=\begin{cases}1 & T>2\\ 0 & otherwise.\end{cases}$$

a) Find and sketch the power function of this test.
b) Find the size and power of the test when testing $H_0:\lambda=1$ versus $H_1:\lambda=4$.
c) Will the test be unbiased for testing:
i) $H_0\lambda\le\lambda_0$ versus $H_1:\lambda>\lambda_0$?
ii) $H_0\lambda=\lambda_0$ versus $H_1:\lambda\ne\lambda_0$?
iii) $H_0\lambda\ge\lambda_0$ versus $H_1:\lambda<\lambda_0$?

Stat 883

1. Let $X_{1,}\ldots X_n$ be a random sample of continuous random variables from a *gamma(*α = 10*,* β) distribution.

a) Show that $Q=\bar{X}/\beta$ is a pivotal quantity for β.

b) Construct a two-sided 90% confidence interval for β.

c) Use the confidence interval to construct a test of $H_0:\beta=2$ versus $H_1:\beta\neq 2$.

d) Find the power function of the test.

2. Suppose that $X_{1,}\ldots X_n$ are iid discrete random variables from a negative binomial(r = 1, p) distribution. Recall that a negative binomial(r, p) has the following properties.

$$f(x|p)=\binom{r+x-1}{x}p^r(1-p)^x;\ \ 0\leq p\leq 1;\ \ x=0,1,2,\ldots$$

$$E(X)=\frac{r(1-p)}{p}$$

$$var(X)=\frac{r(1-p)}{p^2}.$$

a) Derive the MLE of p.

b) Show that $T=\sum_{i=1}^{n} X_i$ is a complete sufficient statistic for p.

c) Using Rao-Blackwell show that the UMVUE of p is $\dfrac{n-1}{n+T-1}$.

d) Verify that your UMVUE of p is unbiased.

e) Estimate p using both the MLE and the UMVUE of p for the following data: 4, 2, 6

f) Show that p has monotone likelihood ratio in $-\sum_{i=1}^{n} X_i$.

g) Derive the level-α UMP test of the $H_0: p\leq p_0$ versus $H_1: p>p_0$. You do **NOT** need to find K.

Stat 883

1. Suppose that $X_{1,} \ldots, X_n$ are iid continuous random variables with the following properties

$$f(x|\lambda)=\frac{2x}{\lambda}\exp(-x^2/\lambda);\;\; x>0;\;\; \lambda>0$$
$$X_i^2 \sim exponential(\lambda)$$
$$E(X^k)=\lambda^{k/2}\Gamma\left(1+\frac{k}{2}\right).$$

a) Derive the MLE of λ.

b) Find the mean and variance of the MLE of λ.

c) Derive the UMVUE of λ.

d) The likelihood ratio test for testing the hypothesis $H_0:\lambda=\lambda_0$ versus $H_1:\lambda>\lambda_0$ is

$$\phi(X)=\begin{cases}1 & \sum X_i^2>C\\ 0 & otherwise\end{cases}$$

Find the exact critical value for a size-α test.

e) Find the power function of the test.

f) Construct a one-sided confidence interval using the hypothesis test from above.

2. Let $X_{1,} \ldots, X_n$ be a random sample of continuous random variables from a distribution with pdf

$$\frac{1}{\lambda}e^{-(x-\mu)/\lambda};\;\; \mu>0;\;\; \lambda>0;\;\; x>\mu.$$

a) Find the density of $X_{(1)}$.

b) Derive a LRT for testing $H_0:\mu=\mu_0$ versus $H_1:\mu\neq\mu_0$. Find an approximate critical value for a size-α test using the large sample approximation.

3. Let $X_{1,} \ldots, X_n$ be a random sample from a $normal(\mu_X, 1)$ and $Y_{1,} \ldots, Y_n$ be an independent random sample from a $normal(\mu_Y, 1)$.

a) Find the distribution of $\bar{X}-\bar{Y}$.

b) Find a pivotal quantity for $\mu_X-\mu_Y$.

c) Derive a 95% confidence interval for $\mu_X-\mu_Y$.

Stat 907

1. State Theorem 7.

2. Prove the invariance property of Theorem 7.

3. Let $y \sim N(X\beta, I\sigma^2), \hat{\beta}=(X'X)^{-}X'y, W=XA,$ and $\hat{\alpha}=(W'W)^{-}W'y.$

a) Show that $X(X'X)^{-}X'W=W.$

b) Find the distribution of $\hat{e}=y-X\hat{\beta}.$

c) Find the distribution of SSH = $y'X\hat{\beta}-y'W\hat{\alpha}.$ For this problem you do not need to find the parameters.

d) Show that SSH and $\hat{e}$ are independent.

4. Let $y_{ij}=\mu+\tau_i+b_i x_j+e_{ij}$ where y_{ij} is the dependent variable, μ is the intercept, τ_i is the effect of treatment *i*, b_i is the linear effect of covariate x_i associated with treatment *i*, and e_{ij} is the residual. The data are summarized in the following table:

Y	Trt	X
y11	1	-1
y12	1	0
y13	1	1
y21	2	-1
y22	2	0
y23	2	1

X = ()

Set up the normal equations.

5. For the rank 3 matrix:

$$A=\begin{pmatrix} 4n & 2n & 2n & 0 & 0 \\ 2n & 2n & 0 & 0 & 0 \\ 2n & 0 & 2n & 0 & 0 \\ 0 & 0 & 0 & 4n & -4n \\ 0 & 0 & 0 & -4n & 4n \end{pmatrix}$$

a) Why is A a rank 3 matrix?

b) Find a generalized inverse of A.

Stat 970

1. Let $Y \sim N_n(X\beta, \sigma^2 I_n)$. Derive the distribution of $X\hat{\beta}$. Be sure to specify all parameters and simplify as much as possible.

2. Show that $B^- A^-$ is a generalized inverse of AB if and only if $A^- ABB^-$ is idempotent.

3. Consider an experiment with two treatments where the experimental units are organized in randomized complete blocks. There are three blocks, and each treatment appears only once in each block.

a) In this experiment, one could test for a treatment effect using an ANOVA F-test. What is a different, equivalent, way of carrying out the test for a treatment effect in this case?

b) Write down X and β.

c) Find X'X and X'y.

4. Let $y \sim N\left(\begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}4 & 2\\2 & 4\end{pmatrix}\right)$. Let $A = \begin{pmatrix}1/2 & -1/2\\-1/2 & 1/2\end{pmatrix}$. What is the distribution of Y'AY (including parameters)? Justify your answer.

Stat 970

1. Let $y_1, \ldots, y_n$ be n uncorrelated random variables with $E(y_i) = i\mu$ and $Var(y_i) = i\sigma^2, i = 1, \ldots, n$. μ and σ^2 are unknown parameters. Find the BLUE of μ.

2. Consider the linear model $y = X\beta + e$ where $y \sim N(X\beta, \sigma^2 V)$ and V is a known, non-singular matrix of constants. y has dimension n x 1. DERIVE the maximum likelihood estimator for β.

3. Consider a 1-way ANOVA type experiment with three treatments. The n observations from treatment 1 have mean $\mu + \alpha_1$. The n observations from treatment 2 have mean $\mu + \alpha_2 + \alpha_3$. The n observations from treatment 3 have mean $\mu + \alpha_2 - \alpha_3$. Note that this is NOT the usual 1-way ANOVA model. Let $\beta = (\mu \ \ \alpha_1 \ \ \alpha_2 \ \ \alpha_3)'$.

a) What is the design matrix X you would use to express this as a linear model?

b) Show that the following define a least squares estimator for β.

$$\begin{pmatrix} \hat{\mu} \\ \hat{\alpha}_1 \\ \hat{\alpha}_2 \\ \hat{\alpha}_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2n} \sum_i \sum_j y_{ij} - \frac{2}{2n} \sum_j y_{1j} \\ \frac{-2}{2n} \sum_i \sum_j y_{ij} + \frac{3}{2n} \sum_j y_{1j} \\ 0 \\ \frac{1}{2n} \left(\sum_j y_{2j} - \sum_j y_{3j} \right) \end{pmatrix}$$

c) Which of the following are estimable?

i) μ	iv) α_3
ii) α_1	v) $\alpha_1 - \alpha_2$
iii) α_2	vi) $2\mu + \alpha_1 + \alpha_2$

d) Pick one estimable function $k'\beta$ in c) and find the standard error of the estimator.

Stat 970

1. Let $y \sim N(X\beta, I\sigma^2)$ and $k'\beta$ be an estimable function. Derive the Best Linear Unbiased Estimator of $k'\beta$.

2. Let $y_{ijk} = \mu + \alpha_i + \beta(x_j - \bar{x}.) + e_{ijk}$ where $i = 1, \ldots, a,\ j = 1, \ldots, b,\ k = 1, \ldots, n$ and the e_{ijk} are iid $N(0, \sigma^2)$. Note $\sum_j (x_j - \bar{x}.) = 0$.

a) Set up and solve the normal equations.

b) Show which of the following are/are not estimable.

i) μ

ii) $\alpha_1 - \alpha_2$

iii) β

3. Let $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \sim N\left(\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}\beta, \begin{pmatrix} I_n\sigma_1^2 & 0 \\ 0 & I_n\sigma_2^2 \end{pmatrix}\right)$. That is the two groups can have different residual variances.

a) Find the distribution of $y_i'(I - X_i(X_i'X_i)^- X_i')y_i$.

b) Find the distribution of $\dfrac{y_1'(I - X_1(X_1'X_1)^- X_1')y_1}{y_2'(I - X_2(X_2'X_2)^- X_2')y_2}$.

c) Construct a two sided 95% confidence interval for σ_1^2/σ_2^2.

d) How could you use the confidence interval to test $H_0: \sigma_1^2 = \sigma_2^2$ versus $H_1: \sigma_1^2 \neq \sigma_2^2$.

Appendix

$$\frac{dx}{dx'} = \frac{dx'}{dx} = I$$

$$\frac{dAx}{dx'} = A$$

$$\frac{dx'B}{dx} = B$$

$$\frac{dx'Ax}{dx} = 2\mathrm{Ax}$$

$$\frac{dx'Ax}{dx'} = 2\mathrm{x}'A$$

$$\frac{du'v}{dx'} = u'\frac{dv}{dx'} + v'\frac{du}{dx'}$$

$$\frac{d\,tr[H]}{dy} = tr\left(\frac{dH}{dy}\right)$$

$$\frac{dV^{-1}}{dy} = -V^{-1}\frac{dV}{dy}V^{-1}$$

$$\frac{d\ln(|V|)}{dy} = tr\left(V^{-1}\frac{dV}{dy}\right)$$

Stat 970

1. Prove that $k'\beta$ is estimable if and only if $k'=k'(X'X)^{-}X'X$.

2. Let $y_1, \ldots, y_n$ be uncorrelated observations with $E(y_i)=a_i\mu$ and $Var(y_i)=a_i\sigma^2$. Here $a_1, \ldots, a_n$ are known positive constants and μ is an unknown parameter.

a) Write $y=(y_1 \ldots y_n)'$ as a linear model. What are X and β?

b) Show that the BLUE of μ is $\hat{\mu}=\frac{\sum_{i=1}^{n} y_i}{\sum_{i=1}^{n} a_i}$.

c) Compute the variance of $\hat{\mu}$.

d) Find the OLS estimator of μ in this model.

3. Suppose we consider two different linear models for the same data y. Write these two models as Model 1: $y=X\beta+e$ and Model 2: $y=W\gamma+e$. We showed this implies that W = XB for some matrix B. Let $\hat{\gamma}$ be a least squares estimator of y in Model 2, and let $\hat{\beta}=B\hat{\gamma}$. It turns out that $\hat{\beta}=B\hat{\gamma}$ is a least squares estimator of β in Model 1.

Now consider this special case. Consider a 1-way ANOVA with 3 treatments, and n_i observations from the i^{th} treatment. Let Model 1 denote the ANOVA model $y_{ij}=\mu+\alpha_i+e_{ij}(i=1,2,3;\ j=1,\ldots,n_j)$. Let Model 2 denote the cell means model for this data.

a) Which of the following are estimable using Model 1?

i) μ

ii) $\mu-\alpha_1$

iii) $\mu+\alpha_2$

iv) $\alpha_1-\alpha_2$

v) $\alpha_1+\alpha_2$

b) What are X, W, β, and γ?

c) It is true that R(X) = R(W). Find a matrix B such that W = XB.

d) What is the least squares estimator $\hat{\gamma}$?

e) Find the least squares estimator $\hat{\beta}$ obtained from $\hat{\beta}=B\hat{\gamma}$.

Stat 970

1. Consider two models $y \sim N(X_0\beta_{0,} I\sigma^2)$ and $y \sim N(X_1\beta_1, I\sigma^2)$ where X_0 is the column space of X_1. Find the distribution with parameters of

$$SSH = y'(M_1 - M_0)y.$$

2. Consider the model $y_{ij} = \tau_i + e_{ij}$ where e_{ij} are independently distributed $N(0, \sigma^2)$, $i = 1, 2$, and $j = 1, \ldots, n_i$.

a) Set up and solve the normal equations for this model.

b) Using the testable functions approach, find the test for testing the null hypothesis that $\tau_1 = \tau_2$.

c) Show that the test in the previous equation is equivalent to rejecting when

$$\left| \frac{\bar{y}_{1.} - \bar{y}_{2.}}{\sqrt{MSE\left(\frac{1}{n_1}\frac{1}{n_2}\right)}} \right| > t_{n_1+n_2-2, 1-\alpha/2}.$$

3. Under the model $y \sim N(X\beta, I\sigma^2)$ with σ^2 known, derive the MLE β of when $K'\beta = 0$. You may assume that $K'\beta$ is estimable and that the inverse of $K'(X'X)^- K$ exists.

Stat 970

1. Let y be distributed $N(X\beta,\sigma^2 I)$. Assume that X is n x p with rank r and is NOT full rank. Let $k'\beta$ be a scalar estimable function.

a) What is the best estimator of $k'\beta$?

b) *Derive* the mean and variance of the best estimator in (a).

c) Prove that $X(X'X)^- X'$ is invariant to the specific generalized inverse used. You may use any *other* result in Theorem 7 without proof.

d) Use the result proved in (c) to show that the overall model SS and error SS are invariant to the form of the generalized inverse used.

e) *Derive* the distribution of the overall model SS (including parameters).

f) Demonstrate that the necessary conditions hold to show that the distribution of the ratio of the model mean square to error mean square has a non-central *F* distribution. Give the parameters of this distribution. What hypothesis is being tested if a central *F* distribution is used?

g) Suppose that the variance of y is $\sigma^2 V$ rather than $\sigma^2 I$ (where V is a known, non-singular matrix.) Does this affect your answers to parts (a) and (b)? If so, how? If not, why not?

2. Let y be distributed $N(X\beta,\sigma^2 I)$. Assume that X is n x p and is NOT full rank.

a) Can we get a unique estimate of β in this situation? If so, write the formula for the estimator of β. If not, *explain* why a unique solution is not possible and briefly describe the usual alternative procedure for obtaining a solution for β.

b) Define *estimable function* (in words – no formulae!).

c) How would you explain to a STAT 802 [or better yet, STAT 412 (undergraduate design of experiments)] student why estimable functions are important?

3. Let $y_{ijk}=\mu+\alpha_i+\beta_j+(\alpha\beta)_{ij}+e_{ijk}$, where $e \sim N(0,\sigma^2 I)$ and i = 1, 2; j = 1, 2, 3; and k = 1, …, n_{ij}. The numbers of observations in each treatment combination are shown below:

	B		
A	1	2	3
1	2	2	4
2	1	1	2

Data were analyzed using the code:

```
proc mixed data = final;
      class A B;
      model y = A|B/htype=1,3 solution;
      lsmeans A|B/e;
run;
```

a) What is the Type I hypothesis for A?

b) What is the Type III hypothesis for A?

c) Show that $\beta_2-\beta_3+(\alpha\beta)_{22}-(\alpha\beta)_{23}$ is estimable.

d) Given the solution

The Mixed Procedure

Solution for Fixed Effects

Effect	A	B	Estimate	Standard Error	DF	t Value	Pr > |t|
Intercept			72.0000	3.9581	6	18.19	<.0001
A	1		0.5000	4.8477	6	0.10	0.9212
A	2		0	.	.	.	.
B		1	6.0000	6.8557	6	0.88	0.4151
B		2	-6.0000	6.8557	6	-0.88	0.4151
B		3	0	.	.	.	.
A*B	1	1	-11.0000	8.3964	6	-1.31	0.2381
A*B	1	2	8.0000	8.3964	6	0.95	0.3775
A*B	1	3	0	.	.	.	.
A*B	2	1	0	.	.	.	.
A*B	2	2	0	.	.	.	.
A*B	2	3	0	.	.	.	.

find the lsmean level 2 of A and the lsmean for level 1 of B.

Appendix

$$\frac{dx}{dx'} = \frac{dx'}{dx} = I$$

$$\frac{dAx}{dx'} = A$$

$$\frac{dx'B}{dx} = B$$

$$\frac{dx'Ax}{dx} = 2\text{Ax}$$

$$\frac{dx'Ax}{dx'} = 2\text{x}'A$$

$$\frac{du'v}{dx'} = u'\frac{dv}{dx'} + v'\frac{du}{dx'}$$

$$\frac{d\,tr[H]}{dy} = tr\left(\frac{dH}{dy}\right)$$

$$\frac{dV^{-1}}{dy} = -V^{-1}\frac{dV}{dy}V^{-1}$$

$$\frac{d\ln(|V|)}{dy} = tr\left(V^{-1}\frac{dV}{dy}\right)$$

Stat 970

1. Let $y \sim N(X\beta, V\sigma^2), K'\beta$ be an estimable function, and $\hat{\beta}=(X'V^{-1}X)^{-}X'V^{-1}y.$

   a) Fund the distribution with parameters of $K'\hat{\beta}$.

   b) Show that SSE and $K'\hat{\beta}$ are independent where $SSE=(y-X\hat{\beta})'V^{-1}(y-X\hat{\beta})$.

2. Let $y_{ijk}=\mu+\alpha_1+\beta_j+(\alpha\beta)_{ij}+e_{ijk}$ where $e \sim N(0, I\sigma^2), i=1\ldots3, j=1\ldots2,$ and $k=1\ldots n_{ij}$. The observations for each of the treatment combinations is given below.

A	B: 1	B: 2
1	$y111, y112$	$y121$
2	$y211$	$y211, y222$
3	$y311$	$y321$

The data are analyzed using the following code.

```
Proc Mixed;
class A B;
Model Y=A|B/htype=1,3;
lsmeans A B A*B/E;
```

   a) What is being estimated by the sample mean for the first level of factor A?

`   b) What is being estimated by the least square mean for the first level of factor A?

   c) What is the Type I hypothesis for A?

   d) What is the Type III hypothesis for A?

   e) Suppose that one of the observations were missing so that the data you had to work is given below.

A	B: 1	B: 2
1	$y111, y112$	$y121$
2		$y211, y222$
3	$y311$	$y321$

For each least square mean for factor A, show that they are/are not estimable.

3. Let $y_{ijk}=\mu+\alpha_i+bx_j+c_ix_j+e_{ijk}$ where i = 1, 2, j = 1, 2, 3, and k = 1, …, n.

   a) Set up the normal equations.

   b) What is the `Proc Mixed` code for fitting this model?

4. Let $y \sim N(X\beta, Ie^{\tau})$. That is the variance is equal to e^{τ}.

   a) Derive the MLE of β and τ.

Stat 970

1. Find the distribution of $K'\hat{\beta}$ where $y \sim N(X\beta, I\sigma^2), \hat{\beta}=(X'X)^{-}X'y,$ and $K'\beta$ is estimable.

2. Let $y_1 \sim N(X\beta, I\sigma_1^2)$ and $y_2 \sim N(X\beta, I\sigma_2^2)$ be n x 1 normal random vectors with different variances.

a) Find the MLE estimators of $\beta, \sigma_1^2,$ and $\sigma_2^2.$

b) Let $SSE_1 = y_1'(I-M)y_1$ and $SSE_2 = y_2'(I-M)y_2.$ Find the distribution with parameters of $SSE_1/SSE_2.$

3. Consider the following model

$$y_{ij} = \mu + \alpha_i + b x_{ij} + c_i x_{ij} + e_{ij}$$

where e_{ij} are iid $N(0,\sigma^2)$. The data are given below:

A	x_{ij}	y_{ij}
1	2	$y11$
1	3	$y12$
1	4	$y13$
2	1	$y21$
2	2	$y22$
2	3	$y23$

a) Set up the normal equations for this model.

b) The following SAS program was run.

```
Proc Mixed;
Class A;
Model Y = A X A*X/htype=1,3;
```

What hypothesis is being tested by the Type I sum of squares for A?

c) What hypothesis is being tested by the Type III sum of squares for A?

d) The generalized inverse of X'X is

$$\begin{pmatrix} 2\frac{1}{3} & -2\frac{1}{3} & 0 & -1 & 1 & 0 \\ -2\frac{1}{3} & 7\frac{1}{6} & 0 & 1 & -2.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0.5 & -0.5 & 0 \\ 1 & -2.5 & 0 & -0.5 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Find the distribution of the BLUE of the estimable function $b+c$.